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Newton's Law of Universal Gravitation

MAYTENSOR

$$F = G\frac{m_1 m_2}{r^2} \tag{1}$$

This is it. Newton's law of universal gravitation. A surprisingly simple equation. Pure and elegant. But what does it actually mean? Let's break it down:

F is the resulting gravitational force between two objects. It is always attractive. Its unit is the newton (N).

F is directly proportional to the masses of the two objects m_1 and m_2 (or, to be more precise, to the product of the two masses). The unit of the masses: kilogram (kg), and kg x kg gives kg^2.

r is the distance between the mass centres of the two objects. (Even though r is often used for the radius, this is not the case here. That a radius will pop up a little later below is a coincidence.) Unit: metre (m).

r is squared and finds itself in the denominator of the equation. With that, F is inversely proportional to the squared distance r^2. Therefore, this is an inverse-square law.

G (called 'big' G) is the gravitational constant. It has the same value throughout the entire universe. At least according to the best available evidence. And because of this all-encompassing purview this law is called Newton's law of *universal* gravitation.

$$G = 6.67 \cdot 10^{-11}\frac{Nm^2}{kg^2} \tag{2}$$

G is not to be confused with g ('little' g), which we will get to know shortly.

The units of G look a bit unwieldy. They can be easily derived from the units of the other parameters that go into the equation. Simply solve it for G using units:

$$N = \frac{G \cdot kg \cdot kg}{m^2} \tag{3}$$

$$N \cdot m^2 = G \cdot kg^2 \tag{4}$$

$$G = \frac{Nm^2}{kg^2} \tag{5}$$

If you prefer SI base units[1], swap the newton N for its definition:

$$1\ N = 1\ kg\frac{m}{s^2} \tag{6}$$

And how does this come about? Simply recall how a force is defined: mass x acceleration.

$$F = ma \tag{7}$$

And the units of m and a are kg and m/s^2, respectively. Therefore, G can also be expressed as:

$$G = kg\frac{m}{s^2}\frac{m^2}{kg^2} = \frac{m^3}{s^2 kg} = m^3 kg^{-1} s^{-2} \tag{8}$$

Newton presented his findings in 1686 to the Royal Society as part of his seminal book 'Philosphiae principia naturalis' (typically called just 'the Principia'), first published in 1687.

Inverse-square law

One of the profound insights was the inverse-square nature of the law of gravitation. This was actually discussed by serveral scientists at the time. Most notably, Robert Hooke[2], besides Newton one of the other titans of science in their time, mentioned in a letter to Newton that he considered the inverse-square principle to be applicable to gravitation. But Newton was the first to publish this train of thought. Newton was known to be divisive figure, and Hooke is reported to have been less than amused.

Why was the time ripe? In fact, the inverse-square law is not as far-fetched as it might appear at first sight. If you have a certain source of some force, and you assume that the force emanates out in three-dimensional space in all directions, you can easily imagine the force going out in ever larger concentric spheres. The surface area of a sphere is:

$$A = 4\pi r^2 \tag{9}$$

Evidently, the surface area increases with increasing radius, and specifically with the squared radius r^2. Therefore, any force going out from the centre gets geometrically 'diluted'. See Figure 1.

[1] International System of Units (SI)
[2] The Robert Hooke of Hooke's law, $F = -kx$, which describes the force needed to act on a spring.

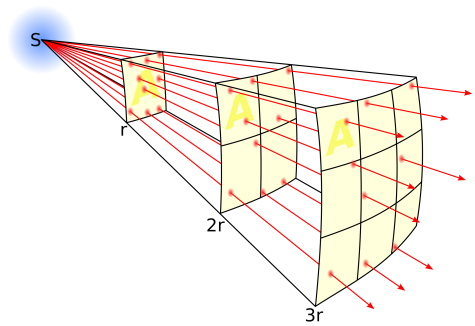

Figure 1: 'Rays' of radiation or force emanate from a source in all directions. As the imagined spheres around the source get larger and larger, the 'density' of the 'rays' decreases proportional to the square of the distance from the source. (Image credit: Borb. CC BY-SA 3.0. https://commons.wikimedia.org/w/index.php?curid=3816716)

As the surface area increases with the squared radius r^2, the intensity of the force going out decreases with the squared radius r^2. And this *decreasing* effect is accommodated for by putting r^2 in the denominator of the equation. Leaving out the specifics of gravitation, the relationship looks like this:

$$intensity = \frac{1}{r^2} \tag{10}$$

The effect of having a square in the denominator is remarkably pronounced:

$$\begin{array}{ccc} r = 1\text{:} & r^2 = 1 & 1/r^2 = 1 \\ r = 2\text{:} & r^2 = 4 & 1/r^2 = 0.25 \\ r = 3\text{:} & r^2 = 9 & 1/r^2 = 0.11 \\ r = 4\text{:} & r^2 = 16 & 1/r^2 = 0.06 \\ \vdots & \vdots & \vdots \end{array}$$

It is even more impressive as a plot (Figure 2):

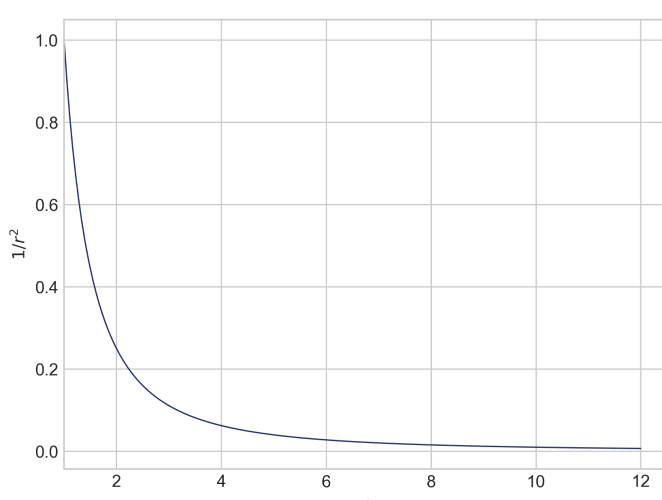

Figure 2: With increasing r, $1/r^2$ falls off precipitously.

And exactly this mechanism applies to the gravitational force. As well as to another law, Coulomb's law (first published in 1785), which looks practically identical and determines the force between two electrically charged particles:

$$F = k\frac{q_1 q_2}{r^2} \tag{11}$$

Big G

With the story of inverse-square law down, we have to tackle G, the gravitational constant already introduced above. It is a proportionality factor that relates the other quantities to each other. As we have seen, its value is:

$$G = 6.67 \cdot 10^{-11}\frac{Nm^2}{kg^2} \tag{12}$$

A decidedly unwieldy number.[3] And the thing is that Newton had absolutely no means of knowing it. It can only be determined empirically. Therefore, someone had to come up with a clever way of measuring it. His name was Henry Cavendish, and he did it more than 100 years after Newton had published his Principia.

The Cavendish experiment

One of the striking features of Newton's law is the fact that gravitation is a property of every piece of mass, applying to everything from the tiniest particle to supermassive black holes (where the gravitational force is so strong that nothing, not even light can escape), and everything in-between. Still, on the scale of everyday (human) life there seems to be only one form of gravitation, that of the Earth acting on objects on its surface. But, in fact, every bit of mass exerts a gravitational force on every bit of other mass, from here to the edge of the observable universe. So why don't we bump into all kinds of physical bodies all the time due to gravitational attraction? Because the force is, apparently, exceedingly small in the usual cases where less than entire planets are involved.

[3] Note that even though G is a universal constant, its number value is not universal at all. It is entirely dependent on the units you want to use. If you prefer parsecs (pc, $1\ pc = 3.0857 \cdot 10^{16}\ m \approx 3.26$ light-years) and solar masses ($M_\odot$) over SI units, then $G = 4.30 \cdot 10^{-3}\ pc \cdot M_\odot^{-1} \cdot (km/s)^2$.

Consider this: With Newton's law being

$$F = G\frac{m_1 m_2}{r^2} \tag{13}$$

and two masses of, say, 100 kg each, and a distance of 1 m between their centres of mass, what is the gravitational force between them? The calculation would be a most simple one if only we knew the value of G. (Please ignore the fact for a moment that now G can be looked up anytime and everywhere. In Newton's time really nobody knew.) But G, as mentioned above, is an empirical constant. It cannot be deduced from other known physical constants. It must be measured. Somehow.

Let's do a thought experiment first. Obviously, the force between the two masses must in any case be tiny. Pick a number. Why not 1 μN (= 10^{-6} N)? This is definitely a really small force that is imperceptible to any of our senses. By putting these numbers into Newton's law and rearranging the equation, we get a most crude estimation of G.

$$G = \frac{Fr^2}{m_1 m_2} = \frac{10^{-6} \cdot 1^2}{100 \cdot 100} = 10^{-10}\frac{Nm^2}{kg^2} \tag{14}$$

But how on Earth can you measure a force on the order of 1 μN? Henry Cavendish had an idea (or, to be more precise, he carried through an idea by geologist John Michell, who died a few years earlier). He took a rod, stuck a weight on each end, attached a wire exactly in the middle of the rod so that the weights were balanced, and suspended everything from the ceiling with the rod and weights a bit above the floor. This weird sounding contraption has even a name. It is called a torsion balance (Figure 3).

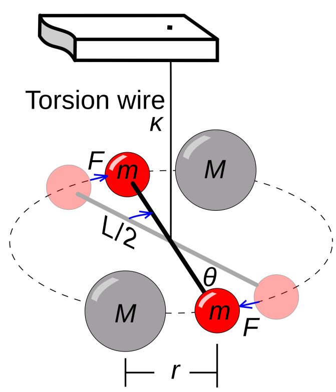

Figure 3: Schematic of a torsion balance as used in the Cavendish experiment. (Image credit: Chris Burks (Chetvorno). Public Domain. https://commons.wikimedia.org/w/index.php?curid=2660162)

First, this pendulum of sorts oscillated considerably, but the motion decreased until it came almost to a standstill. Then, with utmost care as to not disturb the equilibrium unduly, Cavendish moved two additional weights (denoted M in Figure 3) very close to the weights on the rods. If there was a gravitional force between the stationary and the suspended weights, they should attract each other, causing the rod to rotate a little bit. And by measuring this angle of rotation and knowing the stiffness of the wire (resisting the rotation), it is possible to work out the value of the attracting force, and, in due course, as all the other parameters are known entities, the value of G. One can easily imagine how excrutiatingly fiddly the experimental setup must have been. Even the tiniest air draughts or vibrations would disturb the measurement. But Cavendish invested the better part of a year. And he succeeded! He measured the force and worked out... the relative density of the Earth to be 5.448 ± 0.033 times that of water.

Wait. Weren't we supposed to determine G? Yes, we were. It is simply a matter of reformulating the Cavendish result into our now more common terms, which turned out to be:

$$G = 6.74 \cdot 10^{-11} \frac{Nm^2}{kg^2} \tag{15}$$

With that Henry Cavendish got his result right within only 1 % of the currently accepted value. Truly an amazing achievement. (And our own result from the thought experiment above was off only by somewhat over one order of magnitude. Not that bad either.)

Mass of the Earth

Besides his law of gravitation, Newton came up with other smart ideas as well. One is his second law of motion (also published in the Principia):

$$F = ma \tag{16}$$

Force equals mass x acceleration.

Obviously, the law of gravitation and the second law of motion have something in common. Both feature something concerning masses and describe forces. Let's put them side by side:

$$F = G\frac{m_1 m_2}{r^2} \qquad\qquad F = ma \tag{17}$$

Now think of a falling object. Perhaps an apple. Or a banana. It falls due to the force of gravitation and gets continuously accelerated. (We simply ignore air resistance (drag) here.) Diligent experimentalists have established long before Cavendish a fairly good value for this free fall acceleration, which even has its own name, g, or, 'little' g:

$$a = g = 9.81\ ms^{-2} \tag{18}$$

Couldn't we just put the two laws together and see what happens? Say, m is the mass of the banana, and M is the mass of the Earth.

$$mg = G\frac{Mm}{r^2} \tag{19}$$

The banana m cancels out immediately, leaving:

$$g = \frac{GM}{r^2} \tag{20}$$

Where from here? We have values for g, G, and, actually also for r, Earth's radius, 6371 km ($= 6.371 \cdot 10^6$ m)[4], therefore, we can solve for M:

$$r^2 g = GM \tag{21}$$

$$M = \frac{r^2 g}{G} = \frac{(6.371 \cdot 10^6)^2 \cdot 9.81}{6.67 \cdot 10^{-11}} = 5.97 \cdot 10^{24}\ kg \tag{22}$$

We have successfully calculated the mass of the Earth!

Little g

In order to 'weigh' the Earth, above we have put the two law's of Newton side by side. Let's do this again:

$$F = G\frac{m_1 m_2}{r^2} \qquad F = ma \tag{23}$$

On looking closely, we can see that these two laws have the same fundamental structure. Here is what I mean: First, make the variable names a bit clearer by putting Earth's mass into the equation with a capital M and the odd object we are observing with a lower-case m (as we have done above).

$$F = G\frac{Mm}{r^2} \qquad F = ma \tag{24}$$

[4] Please note that we are not so much interested in the radius of the Earth per se. It simply happens to be the distance between the centre of the Earth and its surface. And in Newton's law of gravitation, only the distance matters.

Regroup the left equation:

$$F = \frac{GM}{r^2} m \qquad F = ma \tag{25}$$

$$F = m \frac{GM}{r^2} \qquad F = ma \tag{26}$$

Intriguingly, the term containing the gravitational constant G, Earth's mass M, and Earth's radius r looks like the acceleration a of the second equation. Let's set them equal:

$$a = \frac{GM}{r^2} \tag{27}$$

Now we plug in the known values of G, M and r, and out we get out the gravitational acceleration g:

$$g = \frac{GM}{r^2} = \frac{6.67 \cdot 10^{-11} \cdot 5.97 \cdot 10^{24}}{(6.371 \cdot 10^6)^2} = 9.81 \ ms^{-2} \tag{28}$$

Therefore, the familiar gravitational acceleration g on the surface of the Earth is fully determined by combining the gravitational constant G and the physical characteristics (mass and radius) of the Earth.

Calculating g is fine, as long as we have the mass of the Earth. For reasons of practicality, such a weighing experiment would prove difficult. As outlined above, we have to come from the other direction and *measure* g experimentally instead.

One of history's most notable thinkers before Newton was Galileo Galilei. Among his innumerable other areas of interest, he also thought deeply about the nature of objects accelerating downwards.

An obvious experiment is dropping an object from some known height and measuring the time it took to hit the ground. Easier said than done. Why? Because things tend to fall really fast. Let your cell phone slide from your hand and try to catch it before it smashes into your tiled floor (leaving you with a cracked screen and, to make matters just that bit worse, a cracked tile, too). Hurtling in the direction of the Earth's centre, your phone's speed picks up remarkably briskly. With an acceleration of $g = 9.81 \ ms^{-2}$ (or, in words, 9.81 metres per second per second), this is the velocity after the first few seconds:

$$\begin{aligned} t = 1: &\quad v = 9.81 \ ms^{-1} \\ t = 2: &\quad v = 19.62 \ ms^{-1} \\ t = 3: &\quad v = 29.43 \ ms^{-1} \\ \vdots \quad &\qquad\quad \vdots \end{aligned}$$

Apparently, we have a pattern here. After every additional second, the velocity is higher by 9.81 ms^{-1} (or 35.3 km/h). A nice observation, but we want to derive this relationship more formally.

Consider this:

$$a = g \tag{29}$$

Looks unimpressive. But what is acceleration at its core? It is the rate of change of velocity. If this sounds heavily like plucked directly from differential calculus, you are absolutely right.

$$\frac{dv}{dt} = a \tag{30}$$

With v for velocity and t for time. Therefore:

$$\frac{dv}{dt} = g \tag{31}$$

This is a differential equation. How can we solve it in order to get v? We have to integrate it. Don't panic. In this case here, integration (with respect to t) couldn't be simpler[5]:

$$v = gt \tag{32}$$

And this equation describes exactly the pattern we have shown above: With every second, the velocity increases by the value of g. The table from just above now expanded:

$$\begin{array}{ll} t = 1: & v = gt = 9.81 \cdot 1 = 9.81\ ms^{-1} \\ t = 2: & v = gt = 9.81 \cdot 2 = 19.62\ ms^{-1} \\ t = 3: & v = gt = 9.81 \cdot 3 = 29.43\ ms^{-1} \\ \vdots & \vdots \end{array}$$

While we are at it, let's go one step further. Asking basically the same question as above: What is velocity at its core? It is the rate of change of distance. Therefore, if we integrate velocity, we get distance (s):

$$\frac{ds}{dt} = v \tag{33}$$

$$\frac{ds}{dt} = gt \tag{34}$$

Integrated:

$$s = \frac{gt^2}{2} \tag{35}$$

But what in this world has this to do with our original task of measuring g?

[5] To make things a bit clearer, I deliberately leave out the integration constant.

Everything. As it is technically challenging to measure acceleration directly, time (t) and distance (s) are way simpler to determine with good accuracy and precision. And as soon as we have experimental values for t and s, we only have to plug them into the equation from above after solving it for g:

$$s = \frac{gt^2}{2} \tag{36}$$

$$2s = gt^2 \tag{37}$$

$$g = \frac{2s}{t^2} \tag{38}$$

Although there is still the nagging problem that free falling bodies accelerate really quickly making measurements of time and distance a difficult undertaking. Just think about how far an object will have fallen after only a few seconds. The equation for distance derived above features an element that signifies clearly that things will get out of hand really speedily: Time is squared. Here is a plot of distance over time:

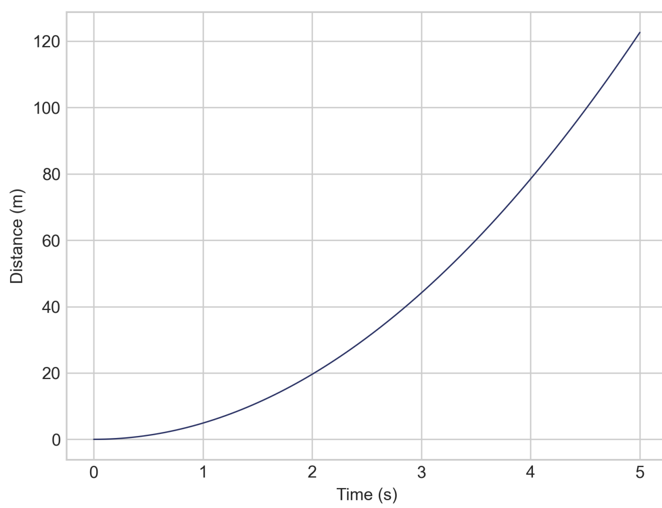

Figure 4: The distance travelled by a free-falling object increases with the square of time t ($s = \frac{gt^2}{2}$).

After only 5 seconds, the falling object has travelled more than 120 metres! If only we could slow things down so that we don't need a skyscraper to toss an apple from. But, by all means, we simply cannot reduce the gravitational acceleration down to more amenable values.

Except, we can.

Galilei's experiment

All we need is a plank of wood and some vectors. Galileo Galilei showed us how 400 years ago.

The trick is this: The gravitational acceleration points straight down, to the centre of the Earth. As such, this acceleration is a vector. It has a magnitude and a direction. (You might already see where we are heading...) Any vector can be seen as the sum of other vectors, or the other way around, a vector can be split up into constituent vectors with different directions. If we do this with gravitational acceleration, we effectively can reduce its magnitude to any value we want, from its nominal 9.8 ms^{-2} right down to 0. We simply have to vary the angle of a 'falling' object from straight down to something a bit more shallow. Ok, such a procedure contradicts the notion of free fall, but what comes decently close to falling is rolling. And here we are: By letting a sphere or a cylinder or something of similar shape roll down an inclined plane (e. g. a plank of wood), we can redirect the gravitational pull and reduce it so that measuring time and distance is much easier to achieve. Do this with care in a controlled setup, and you can determine the value of Earth's gravitational acceleration empirically: Galileo Galilei's inclined plane experiment. Let's recreate it.

A fairly flat piece of wood that can be positioned at any desired angle is an IKEA shelf board. As moving object I use a bolt with two washers acting as flanges on each end of the bolt (Figure 5).[6]

Figure 5: This is my rolling stock, a 70 mm long bolt with a washer attached to each end. Rolls remarkably well.

[6] I have tried other cylindrical objects as well, but none of them came close to the bolt-washer solution. Specifically, an aluminium bottle, quite precisely manufactured and rolling really smoothly, turned out to be much more of the sluggish type. It rolled considerably slower, most probably due to a relevant amount of air resistance.

Distances are easily determined with a tape measure. So far, so good. But how about getting precise time readings? You certainly could fit some light gate sensors to the board and record the instances the object passes them using, say, a Raspberry Pi microcontroller. (Thinking of it, I will definitely do this at some point.) But there is a far simpler method. Use a camera and make a video of the rolling object. With this (and a video editing software, e. g. DaVinci Resolve) you can precisely determine where the object is at any time point, frame by frame.

Here is a schematic of my setup's geometry with all the necessery dimensions:

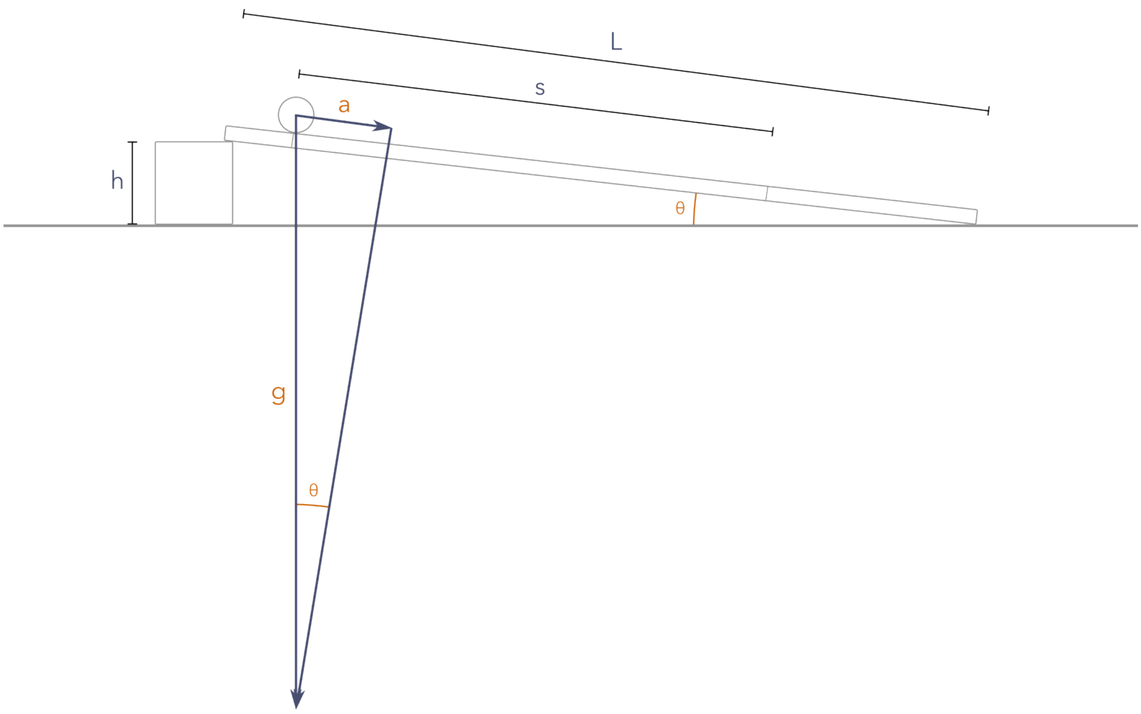

Figure 6: L is the length of the board, and h the height of the block supporting the board on one end. Given these two parameters, we can calculate the angle θ (theta). s is the distance for which we record the time it takes for the object to roll. The parameters a and g are calculated.

But how do we get from here to a value of g? First, we work out the value of the acceleration of the object as it rolls down the board. To do this we need the angle θ.

$$L = 0.96\ m;\ \ h = 0.041\ m \tag{39}$$

With this we can calculate θ. Time to get out some of the old and rusty trigonometry. Remember SOH-CAH-TOA? As L, h and θ form a right triangle, and with h being the opposite and L the hypotenuse, we summon the sine here:

$$sin(\theta) = \frac{h}{L} \tag{40}$$

To get the actual value of θ:

$$\theta = arcsin\left(\frac{h}{L}\right) = arcsin\left(\frac{0.041}{0.96}\right) = 0.0427\ radians = 2.447\ degrees \tag{41}$$

Now we are ready to do the measurements. Place the flanged bolt carefully at the start mark and let it roll down the inclined plane with the camera recording this experimental spectacle. (I used the highest frame rate my gear could handle, 60 frames per second. Higher would be better.) Repeat several times. In the video editing software, it is simple to extract the time stamps, first when releasing the bolt and then when it passes the second mark positioned at 40 cm. Calculate all the individual rolling time intervals and then their mean (t_{mean}). Here is my result (after four valid measurements):

$$t_{mean} = 1.4\ s \tag{42}$$

In the previous section, we have already derived the equation that describes the distance over time by integrating first acceleration and then velocity. Here again, in short:

$$a = g \tag{43}$$

$$v = gt \tag{44}$$

$$s = \frac{gt^2}{2} \tag{45}$$

As we have the distance and the time, we can calculate the acceleration (let's call it a here, because g, in the sense of little g, is what we want to find out.

$$s = \frac{at^2}{2} \tag{46}$$

$$2s = at^2 \tag{47}$$

$$a = \frac{2s}{t^2} \tag{48}$$

$$a = \frac{2 \cdot 0.4}{1.4^2} = 0.4082\ ms^{-2} \tag{49}$$

Now for the crucial part. We have the actual acceleration of the flanged bolt rolling down the inclined plane. How can we get the corresponding value of g?

Look again at the figure above. In addition to the first right triangle (concerning the board), we have a second one formed by the acceleration a, gravitational acceleration g and the angle θ. (Yes, geometry has it that this is exactly the same angle θ that we have observed between the flat 'ground' and the inclined plane. Very handy.) The rest is practically a walk in the park. Simply calculate g in its capacity of being the hypotenuse in this right triangle.

$$sin(\theta) = \frac{a}{g} \tag{50}$$

$$g = \frac{a}{sin(\theta)} \tag{51}$$

$$g = \frac{0.4082}{sin(0.0427)} = 9.563\ ms^{-2} \tag{52}$$

And this result, 9.563 ms^{-2}, I find is a pretty good approximation of the true value of 9.8 ms^{-2}.

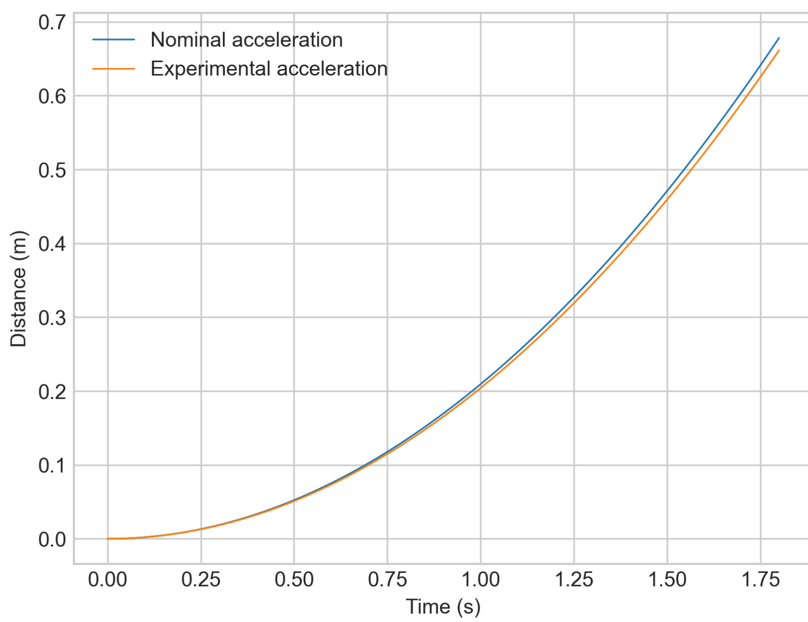

Figure 7: Comparison of nominal acceleration based on the true value of g and our experimental result by plotting the rolling distance over time.

Experimental physics on the kitchen table.

It is worth reitering that an object rolling down an inclined plane is, in principle, the same as the object being in free fall. The mechanism and the equations are exactly the same. Only the acceleration is reduced from its maximum of $g = 9.8\ ms^{-2}$ to a lower value depending on the angle of inclination.

Surface gravity

Now we have everything neatly derived. Cavendish determined big G, we confirmed little g. In addition, we have shown the close relationship between two of Newton's most important laws, the law of gravitation and the second law of motion:

$$F = \frac{Gm_1m_2}{r^2} \qquad F = ma \tag{53}$$

If we wanted to calculate the gravitational force acting on a mass of 1 kg, we could use both laws and get the same answer:

$$F = \frac{Gm_1m_2}{r^2} = \frac{6.67 \cdot 10^{-11} \cdot 5.97 \cdot 10^{24} \cdot 1}{(6.371 \cdot 10^6)^2} = 9.8\ N \tag{54}$$

Or simpler:

$$F = ma = 1 \cdot 9.8 = 9.8\ N \tag{55}$$

Please ponder for a moment how tiny this force is. Whereas the mass of the entire Earth ($5.97 \cdot 10^{24}\ kg$) is needed to pull down a mass of 1 kg by way of gravitation, you can most easily overcome this force with your arm, simply by lifting 1 kg up. Earth's mass vs. your arm. And your arm wins. Pretty amazing.

As Newton's law of gravitation is universal (as is G), it is a simple task to calculate the gravitational force exerted on a mass of 1 kg on another celestial body, say, the Moon (with a mass of $7.342 \cdot 10^{22}\ kg$ and a radius of $1.737 \cdot 10^6\ m$):

$$F = \frac{Gm_1m_2}{r^2} = \frac{6.67 \cdot 10^{-11} \cdot 7.342 \cdot 10^{22} \cdot 1}{(1.737 \cdot 10^6)^2} = 1.6\ N \tag{56}$$

The Moon's gravity on its surface is about $1.6/9.8 \approx 1/6$ that of the Earth.

Surface gravity depends, crucially, on the radius of the planet or moon in question as the radius determines how close another object can come to its centre of mass. Let's make a thought experiment here. What if we compacted the Earth down to the density of a neutron star? The mass would remain the same, but the radius, and with that the distance to an object on its surface, would be much shorter.

A good neutron star has a density of

$$d = 4.8 \cdot 10^{17}\ kg\, m^{-3} \tag{57}$$

Compare that with Earth's meager density of about $5,500\ kg\, m^{-3}$ ($= 5.5 \cdot 10^3\ kg\, m^{-3}$, or 14 orders of magnitude less than the neutron star).

Density is simply mass divided by volume:

$$d = \frac{m}{V} \tag{58}$$

Therefore, given mass and density, we can rearrange the equation for V:

$$V = \frac{m}{d} \tag{59}$$

With the values we already know, a compacted Earth has a volume of:

$$V = \frac{5.97 \cdot 10^{24}}{4.8 \cdot 10^{17}} = 1.24 \cdot 10^{7}\ m^{3} \tag{60}$$

The volume of a sphere is:

$$V = \frac{4}{3}\pi r^{3} \tag{61}$$

Solved for the radius:

$$r = \sqrt[3]{\frac{3V}{4\pi}} \tag{62}$$

And with the new volume of the Earth plugged in:

$$r = \sqrt[3]{\frac{3 \cdot 1.24 \cdot 10^{7}}{4\pi}} = 143.6\ m \tag{63}$$

Crushed down to the density of a neutron star, the entire Earth would have a radius of only 143.6 m! Consider this for a moment! What would the surface gravity be (for 1 kg of mass)?

$$F = G\frac{m_1 m_2}{r^2} = 6.67 \cdot 10^{-11}\frac{5.97 \cdot 10^{24} \cdot 1}{(143.6)^2} = 1.93 \cdot 10^{10}\ N \tag{64}$$

Same mass, much shorter distance, and the surface gravity shoots up from our standard 9.8 N to more than 19 billion N. Pretty inhospitable there. But a direct consequence of the inverse square law in action.

Gravity at a distance

Right from the start we stated that every bit of mass in this universe exerts a gravitational force on every other bit of mass. As humans we are plainly unable to sense anything of that directly apart from the usual pull of Earth's gravity. Let's check this observation. Imagine yourself standing on the high street of Stratford, New Zealand. Looking to the west you see Mount Taranaki:

Figure 8: Mount Taranaki, North Island, New Zealand. A pretty symmetrical cone-shaped volcano, which makes calculating its mass a lot easier (the hump on its left flank in the image notwithstanding). It is worth looking it up on Google Maps. An impressive sight. (Image credit: Public domain. https://en.wikipedia.org/wiki/Mount_Taranaki#/media/File:Mt_Taranaki.JPG)

What is the amount of gravitational force with which Mount Taranaki attracts you (and vice versa)?

We can approximate the volume (V) of Mount Taranaki as that of a cone:

$$V = \frac{r^2 \pi h}{3} \tag{65}$$

With estimates for the radius at the base ($r = 7000\ m$) and a prominence above the plain as height $h = 2300\ m$:

$$V = \frac{7000^2\ \pi\ 2300}{3} = 1.18 \cdot 10^{11}\ m^3 \tag{66}$$

Then we assume a rock density of $\rho = 3000\ kg\ m^{-3}$ to get the mass:

$$m = V \cdot \rho = 1.18 \cdot 10^{11} \cdot 3000 = 3.54 \cdot 10^{14}\ kg \tag{67}$$

With an approximated distance between Stratford and the mountains centre of gravity of 20 km and you having 70 kg, everything is ready to be plugged into our law of gravitation:

$$F = G\frac{m_1 m_2}{r^2} = 6.67 \cdot 10^{-11} \frac{3.54 \cdot 10^{14} \cdot 70}{20000^2} = 0.0041\ N \tag{68}$$

Even though Mount Taranaki comes with the hefty mass of 354 million million kg (no typo), it exerts only the force of a measly 0.0041 N on you (equivalent to the weight of something with the mass of $0.0041/9.8 = 0.00042\ kg = 0.42\ g$). This is the reason why you don't have to worry about getting knocked over when coming close to even a large mountain.

In contrast, on the celestial stage, the involved forces tend to be somewhat stronger. Take the Earth-Moon system as an example. As it is the case for every moving object, the Moon would shoot off into the vastness of space in a straight line if the Earth would not force it into its orbit by way of gravitation[7]:

$$F = \frac{G m_{Earth} m_{Moon}}{r^2} = \frac{6.67 \cdot 10^{-11} \cdot 5.97 \cdot 10^{24} \cdot 7.34 \cdot 10^{22}}{384,400,000^2} = 1.98 \cdot 10^{20}\ N \tag{69}$$

Unsurprisingly, this is an unfathomable amount of force that is needed to keep the Moon in its tracks in Earth's orbit. But the Earth is not the only celestial body in the vicinity. What is the gravitational force exerted by the Sun on the Moon? Ok, the Sun is vastly heavier than the Earth, but it is also much farther away. Can it have any effect? (For our purposes here, we can approximate the Sun-Moon distance with the Sun-Earth distance.)

$$F = \frac{G m_{Sun} m_{Moon}}{r^2} = \frac{6.67 \cdot 10^{-11} \cdot 1.98 \cdot 10^{30} \cdot 7.34 \cdot 10^{22}}{(1.496 \cdot 10^{11})^2} = 4.33 \cdot 10^{20}\ N \tag{70}$$

It can. The gravitational force between the Sun and the Moon is even a bit greater than between the Earth and its dear companion. After all, the Sun has to keep them both in its orbit.

[7] Please note that the Moon, despite its impressive size in the sky, has only about 1.2 % of Earth's mass.

Conclusion

Gravitation is the dominating force of the universe. Every objects attracts each and every other object, determining the structure of galaxies and keeping our feet on the ground. Galilei and later Cavendish provided empirical data, and Isaac Newton pulled all together into such a deceptively simple but at the same time most powerful equation, Newton's law of universal gravitation:

$$F = G\frac{m_1 m_2}{r^2} \tag{71}$$

Until a very smart person comes along and tells us that gravitation is not a force after all. Gravitation, in this alternative view, is more a deformation of space and time (bundled up as spacetime). This model goes by the grand name of general relativity, and the wizzard who had all these insights was, you knew it, Albert Einstein. But don't throw out all the preceding chapters. Even though general relativity was confirmed ever since its publication in 1915, Newton's law of gravitation is still the valid model in most of the circumstances, at least as long as you travel considerably slower than the speed of light and avoid the occasional black hole in your path.

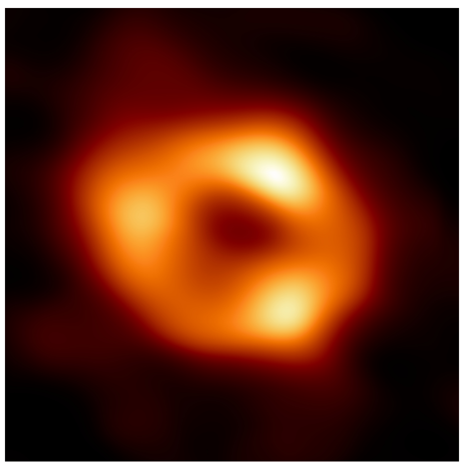

Figure 9: Sagittarius A, the supermassive black hole at the centre of our galaxy. (Image credit: EHT Collaboration. https://www.eso.org/public/images/eso2208-eht-mwa/, retrieved 28 December 2024)

Further reading

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Back matter

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Newton's Law of Universal Gravitation

MAYTENSOR

Hallein, Austria

First edition, December 2024

Published on maytensor.org

Cite as: Gamsjäger T. Newton's law of universal gravitation. Maytensor 2024

This article is part of the Explainer series, in which the aim is to provide explanation, insights and perhaps (or hopefully) some fresh perspectives on selected topics in science and engineering. As standard textbook knowledge is the basis, typically no specific references are given. But selected sources are listed in the 'Further reading' section.

Cover image: 'Earthrise', taken by astronaut Bill Anders on 24 December 1968 during the Apollo 8 mission, the first spaceflight that reached lunar orbit. Public domain.

Typeset with LaTeX.